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Two positive solutions for second-order quasilinear differential equation boundary value problems with sign changing nonlinearities☆

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Abstract

In this paper, the second order quasilinear differential equation $(\Phi(y'))' + q(t)f(t,y) = 0,\ 0 < t < 1$ subject to Dirichlet boundary conditions and mixed boundary conditions is studied, where f is allowed to change sign, $\Phi(v) = |v|^{p-2}v,\ \ p > 1$. We show the existence of at least two positive solutions by using a new fixed point theorem in cones.

Keywords: Quasilinear differential equation; Positive symmetric solution; Fixed point theorem in cones

1. Introduction

In this paper we shall consider the Dirichlet boundary value problem with a p-Laplacian operator

$$\begin{aligned}&(\Phi(y'))' + q(t)f(t,y) = 0,\ \ 0 < t < 1,\\ &\quad y(0) = y(1) = 0\end{aligned} \tag{1}$$

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and the mixed boundary value problem with a p-Laplacian operator

$$\begin{aligned} &(\Phi(y'))' + q(t)f(t,y) = 0,\ 0 < t < 1, \\ &\quad y(0) = y'(1) = 0, \end{aligned} \tag{2}$$

where our nonlinear term f is allowed to change sign and $\Phi(v) = |v|^{p-2}v, p > 1$. The following conditions will be assumed throughout:

(H_1) $f : [0,1] \times [0,\infty) \to [-M,\infty)$ is continuous and $f(t,0) \geqslant 0 (\not\equiv 0)$ for $t \in [0,1]$, where $M > 0$ is a constant;
(H_2) $q : (0,1) \to [0,\infty)$ is continuous and $q(t) \not\equiv 0$ on any subinterval of $(0,1)$, and $\int_0^1 q(t)\,dt < \infty$.

In the recent years, there has been increasing interest in questions of positive solutions of boundary value problems for ordinary differential equations [1,2,4,5,8,9,12,14], as well as for finite difference equations. The recent book by Agarwal et al. [1] gives a good overview of much of the work which has been done and the methods used.

In [9], Erbe and Wang studied the existence of positive solutions of the equation $u'' + q(t)f(u) = 0$ by using the Krasnosel'skii fixed point theorem [10], where $q(t)$ is continuous on $[0,1]$ and $f(u)$ is continuous on $[0,\infty)$. Since then, the Krasnosel'skii fixed point theorem has been widely used to discuss the existence of positive solutions for boundary value problems. For the second-order boundary value problem

$$\begin{aligned} &y'' + f(y) = 0,\ 0 \leqslant t \leqslant 1, \\ &\quad y(0) = y(1) = 0, \end{aligned} \tag{3}$$

where f: $R \to [0,\infty)$ is continuous, Avery [2] imposed conditions on f to yield at least three symmetric positive solutions to (3) by applying the Leggett–Williams fixed point theorem [13]. Avery and Henderson [5] improved Avery's results by using a five functionals fixed point theorem [3]. On the other hand, Avery et al. [4] applied a twin fixed point theorem to obtain at least two positive solutions for the right focal boundary value problem

$$\begin{aligned} &y'' + f(y) = 0,\ 0 \leqslant t \leqslant 1, \\ &\quad y(0) = y'(1) = 0, \end{aligned} \tag{4}$$

where $f : R \to [0,\infty)$ is continuous.

Using the fixed point theorem in cones due to Krasnosel'skii, Junyu Wang [14] studied the existence of positive solutions of the quasilinear differential equation

$$(g(u'))' + q(t)f(u) = 0,\ 0 < t < 1$$

subject to nonlinear boundary conditions, where $g(v) = |v|^{p-2}v,\ \ p > 1$.

In order to apply the concavity of solutions in the proofs, all the above results were done under the assumption that function f is nonnegative. For the sign changing nonlinearity f, few results were done. Motivated by the results mentioned above, in this paper we apply a new fixed point theorem in cones to obtain at least two positive solutions of problems (1) and (2). Growth conditions are imposed on f in later sections so that our fixed point theorem in cones is applicable.

2. The fixed point theorem in cones

For a cone K in a Banach space X with norm $\|\cdot\|$ and a constant $r > 0$, let $K_r = \{x \in K : \|x\| < r\}$, $\partial K_r = \{x \in K : \|x\| = r\}$. Suppose $\alpha : K \to R^+$ is a continuous functional, let

$$K(b) = \{x \in K : \alpha(x) < b\}, \partial K(b) = \{x \in K : \alpha(x) = b\}$$

and $K_a(b) = \{x \in K : a < \|x\|, \alpha(x) < b\}$. The origin in X is denoted by θ.

Definition 1. Given a cone K in a real Banach space X, a functional $\alpha : K \to R$ is said to be concave functional on K provided

$$\alpha(tx + (1-t)y) \geqslant t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in K$ and $0 \leqslant t \leqslant 1$.

Theorem 1. *Let X be a real Banach space with norm $\|\cdot\|$ and $K \subset X$ a cone. Suppose $T : K \to K$ is a completely continuous operator, and $\alpha : K \to R^+$ a continuous concave functional satisfying $\alpha(x) \leqslant \|x\|$ for all $x \in K$. If there are constants $c > b > a > 0$ such that*

(C_1) $\|Tx\| < a$ *for* $x \in \partial K_a$;
(C_2) $\alpha(Tx) > b$ *for* $x \in \partial K(b)$;
(C_3) $\alpha(x) \leqslant b$ *implies* $\|Tx\| < c$,

then T has in K at least a fixed points y such that

$$a < \|y\| < c, \quad \alpha(y) < b.$$

Proof. Let the symbol $\deg_K$ denote the degree on cone K [6]. Then condition (C_1) implies

$$\deg_K \{I - T, K_a, \theta\} = 1.$$

Let $\Omega = K(b) \cap K_{2c}$, we now prove that

$$\deg_K \{I - T, \Omega, \theta\} = 0.$$

Conditions (C_2) and (C_3) imply $\inf_{x\in\partial K(b)} \alpha(Tx) \geqslant b > 0$ and $\inf_{x\in\partial K(b)} \|Tx\| \leqslant c$. Let $\tilde{T} : \bar{K}(b) \to K$ be an extension of $T|_{\partial K(b)} : \partial K(b) \to K$. Dugundji extension theorem [7] ensures that $\tilde{T}$ is completely continuous and $\tilde{T}(\bar{K}(b)) \subset \overline{convex}\, T(\partial K(b))$. Since $\{x \in K : \alpha(x) \geqslant b\} \cap \{x \in K : \|x\| \leqslant c\}$ is a convex set, we have

$$\inf_{x\in\bar{K}(b)} \alpha(\tilde{T}x) \geqslant b > 0 \quad \text{and} \quad \inf_{x\in\bar{K}(b)} \|\tilde{T}x\| \leqslant c.$$

We claim

$$\deg_K \{I - \tilde{T}, \Omega, \theta\} = 0.$$

Clearly $\partial\Omega = (\partial K(b) \cap \bar{K}_{2c}) \cup (\partial K_{2c} \cap \bar{K}(b))$. For $x \in \partial\Omega$, $(I - \tilde{T})(x) \neq \theta$. If it is not true, then there exists $x_0 \in \partial\Omega$ such that

$$x_0 = \tilde{T}x_0.$$

If $x_0 \in \partial K(b) \cap \bar{K}_{2c}$, then

$$b = \alpha(x_0) = \alpha(\tilde{T}x_0) = \alpha(Tx_0) > b,$$

a contradiction. On the other hand, if $x_0 \in \partial K_{2c} \cap \bar{K}(b)$, then

$$2c = \|x_0\| = \|\tilde{T}x_0\| \leqslant c,$$

a contradiction. For $x \in \Omega$, we have

$$\alpha(x) < b \quad \text{and} \quad \alpha(\tilde{T}x) \geqslant b.$$

Thus, $(I - \tilde{T})(x) \neq \theta$ for $x \in \Omega$. It follows that

$$\deg_K\{I - \tilde{T}, \Omega, \theta\} = 0.$$

Take a homotopy $H(x, \lambda) = \lambda Tx + (1 - \lambda)\tilde{T}x$. It is easy to see that

$$H(x, \lambda) \neq x \quad \text{for all } x \in \partial\Omega, \lambda \in [0, 1].$$

Thus,

$$\deg_K\{I - T, \Omega, \theta\} = \deg_K\{I - \tilde{T}, \Omega, \theta\} = 0.$$

From $\alpha(x) \leqslant \|x\|$, we have $K_a \subset K(b) \cap K_{2c} = \Omega$. Then

$$\begin{aligned}
&\deg_K\{I - T, \Omega \setminus K_a, \theta\}\\
&\quad = \deg_K\{I - T, \Omega, \theta\} - \deg_K\{I - T, K_a, \theta\}\\
&\quad = -1.
\end{aligned}$$

So T has in K a fixed point y such that

$$a < \|y\| < c, \quad \alpha(y) < b.$$

Theorem 1 is now proved. □.

3. Dirichlet boundary problem

In this section, we impose growth conditions on f which allow us to apply Theorem 1 to obtain two positive solutions of (1). Let $X = C[0, 1]$, $K = \{x \in X : x(t) \geqslant 0, x(t) = x(1 - t) \text{ for } t \in [0, 1]\}$, $K' = \{x \in K : x \text{ is concave on } [\delta/2, 1 - \delta/2]\}$, where $\delta \in (0, \frac{1}{2})$. Obviously, $K, K' \subset X$ are two cones with $K' \subset K$.

We suppose the following conditions are satisfied:

(H$_3$) $f(t, x), q(t)$ are symmetric about $t = \dfrac{1}{2}$;

and there are $a, b, d > 0$ satisfying

$$0 < \frac{1 - \delta}{2}\Phi^{-1}\left[\Phi\left(\frac{2d}{\delta}\right) + M\int_0^{\delta/2} q(t)\,\mathrm{d}t\right] + d \leqslant a < \delta b < b$$

such that

(H_4) $f(t,x)\int_{\delta/2}^{1/2} q(t)\,\mathrm{d}t \geqslant M\int_0^{\delta/2} q(t)\,\mathrm{d}t$ for $(t,x)\in[\delta/2, 1-\delta/2]\times[d,b]$;

(H_5) $f(t,x)\int_0^{1/2} q(t)\,\mathrm{d}t < \Phi(2a)$ for $(t,x)\in[0,1]\times[0,a]$;

(H_6) $f(t,x)\int_{\delta}^{1/2} q(t)\,\mathrm{d}t \geqslant M\int_0^{\delta} q(t)\,\mathrm{d}t + \Phi(b)$ for $(t,x)\in[\delta, 1-\delta]\times[\delta b,b]$;

(H_7) $f(t,x)\int_0^{\delta/2} q(t)\,\mathrm{d}t \leqslant \Phi(2a/\delta)$ for $(t,x)\in[0,\delta/2]\times[0,b]$.

For $x\in K$ we define

$$\alpha(x) = \min_{\delta\leqslant t\leqslant 1-\delta} x(t)$$

$$(Tx)(t) = \begin{cases} \left(\int_0^t \Phi^{-1}\left(\int_s^{1/2} q(u)F(u,x(u))\,\mathrm{d}u\right)\,\mathrm{d}s\right)^+, 0\leqslant t\leqslant \frac{1}{2}, \\ \left(\int_t^1 \Phi^{-1}\left(\int_{1/2}^s q(u)F(u,x(u))\,\mathrm{d}u\right)\,\mathrm{d}s\right)^+, \frac{1}{2}\leqslant t\leqslant 1, \end{cases}$$

where $(B)^+ = \max\{B,0\}$, $F(t,u) = f(t,u)$ for $u\geqslant 0$ and $F(t,u) = f(t,0)$ for $u<0$.

$$(Ax)(t) = \begin{cases} \int_0^t \Phi^{-1}\left(\int_s^{1/2} q(u)F(u,x(u))\,\mathrm{d}u\right)\,\mathrm{d}s, 0\leqslant t\leqslant \frac{1}{2}, \\ \int_t^1 \Phi^{-1}\left(\int_{1/2}^s q(u)F(u,x(u))\,\mathrm{d}u\right)\,\mathrm{d}s, \frac{1}{2}\leqslant t\leqslant 1. \end{cases}$$

For $x\in X$, define $\theta: X\to K$ by $(\theta x)(t) = \max\{x(t),0\}$, then $T = \theta\circ A$. For $x\in K'$, let

$$(T'x)(t) = \begin{cases} \int_0^t \Phi^{-1}\left(\int_s^{1/2} q(u)f^*(u,x(u))\,\mathrm{d}u\right)\,\mathrm{d}s, 0\leqslant t\leqslant \frac{1}{2}, \\ \int_t^1 \Phi^{-1}\left(\int_{1/2}^s q(u)f^*(u,x(u))\,\mathrm{d}u\right)\,\mathrm{d}s, \frac{1}{2}\leqslant t\leqslant 1, \end{cases}$$

where

$$f^*(t,x) = \begin{cases} f(t,0), & 0\leqslant t<\frac{\delta}{2} \text{ or } 1-\frac{\delta}{2}<t\leqslant 1, -\infty<x<0, \\ f(t,x), & \frac{\delta}{2}\leqslant t\leqslant 1-\frac{\delta}{2}, d\leqslant x\leqslant b, \\ f(t,d), & \frac{\delta}{2}\leqslant t\leqslant 1-\frac{\delta}{2}, -\infty<x<d, \\ f(t,x), & 0\leqslant t<\frac{\delta}{2} \text{ or } 1-\frac{\delta}{2}<t\leqslant 1, 0\leqslant x\leqslant b, \\ f(t,b), & 0\leqslant t\leqslant 1, b<x<+\infty. \end{cases}$$

Lemma 1 (Guo Yanping et al. [11]). *Suppose $A : K \to X$ is completely continuous, then $\theta \circ A : K \to K$ is also completely continuous.*

Condition (H_3) implies that A and T' are well defined. From the continuity of f, it is easy to see that $A : K \to X$ is completely continuous. So $T : K \to K$ is completely continuous by using Lemma 1. For $x \in K'$, we have $x(t) \geqslant \delta/(1-\delta)\max_{\delta/2 \leqslant t \leqslant 1-\delta/2} x(t) \geqslant \delta \max_{\delta/2 \leqslant t \leqslant 1-\delta/2} x(t)$ for $t \in [\delta, 1-\delta]$ by the concavity of x on $[\delta/2, 1-\delta/2]$. Thus, $\alpha(x) \leqslant \|x\|$ and $\alpha(x) \geqslant \delta \max_{\delta/2 \leqslant t \leqslant 1-\delta/2} x(t)$.

Lemma 2. *Suppose* (H_1)–(H_4) *hold. Then $T' : K' \to K'$ is completely continuous.*

Proof. For all $x \in K'$, from (H_1) and (H_4), we have

$$\begin{aligned}
\int_s^{1/2} q(u) f^*(u, x(u))\,\mathrm{d}u &= \int_s^{\delta/2} q(u) f^*(u, x(u))\,\mathrm{d}u + \int_{\delta/2}^{1/2} q(u) f^*(u, x(u))\,\mathrm{d}u \\
&\geqslant -M \int_0^{\delta/2} q(u)\,\mathrm{d}u + \int_{\delta/2}^{1/2} q(u) \cdot \frac{M \int_0^{\delta/2} q(t)\,\mathrm{d}t}{\int_{\delta/2}^{1/2} q(t)\,\mathrm{d}t}\,\mathrm{d}u \\
&= 0 \ \text{ for } 0 \leqslant s \leqslant \frac{\delta}{2}, \\
\int_s^{1/2} q(u) f^*(u, x(u))\,\mathrm{d}u &\geqslant 0 \ \text{ for } \frac{\delta}{2} \leqslant s \leqslant \frac{1}{2},
\end{aligned}$$

thus,

$$\begin{aligned}
&(T'x)(t) = \int_0^t \Phi^{-1}\left(\int_s^{1/2} q(u) f^*(u, x(u))\,\mathrm{d}u\right) \mathrm{d}s \geqslant 0 \ \text{ for } 0 \leqslant t \leqslant \frac{1}{2}, \\
&(\Phi((T'x)'))'(t) = -q(t) f^*(t, x(t)) \leqslant 0 \ \text{ for } t \in \left[\frac{\delta}{2}, 1 - \frac{\delta}{2}\right].
\end{aligned}$$

So $T' : K' \to K'$. Using the continuity of f and the definition of f^*, it is easy to see that $T' : K \to K'$ is completely continuous. □

Theorem 2. *Suppose* (H_1)–(H_7) *hold. Then the boundary value problem* (1) *has at least two positive solutions y_1 and y_2 such that*

$$0 < \|y_1\| < a < \|y_2\|, \ \min_{t \in [\delta, 1-\delta]} y_2(t) < \delta b.$$

Proof. At first we show that T has a fixed point $y_1 \in K$ with $0 < \|y_1\| < a$. In fact, for all $x \in \partial K_a$, we have $\|x\| = a$. From (H_5) we obtain

$$\begin{aligned}
\|Tx\| &= \max_{0 \leqslant t \leqslant 1/2} \left| \left[\int_0^t \Phi^{-1}\left(\int_s^{1/2} q(u) F(u, x(u))\,\mathrm{d}u \right) \mathrm{d}s \right]^+ \right| \\
&= \max_{0 \leqslant t \leqslant 1/2} \max\left\{ \int_0^t \Phi^{-1}\left(\int_s^{1/2} q(u) f(u, x(u))\,\mathrm{d}u \right) \mathrm{d}s, 0 \right\}
\end{aligned}$$

$$< \frac{1}{2}\,\Phi^{-1}\left(\int_0^{1/2} q(u)\cdot\frac{1}{\int_0^{1/2} q(t)\,\mathrm{d}t}\Phi(2a)\,\mathrm{d}u\right)$$
$$= \frac{1}{2}\cdot 2a = a.$$

The existence of y_1 is proved by using the Schauder fixed point theorem.

Obviously, y_1 is a solution of (1) if and only if y_1 is a fixed point of A. Next we will prove that y_1 is a fixed point of A. Suppose this is not true, then there is $t_0\in(0,1)$ such that $y_1(t_0)\neq(Ay_1)(t_0)$. It must be $(Ay_1)(t_0)<0=y_1(t_0)$. Let (t_1,t_2) be the interval such that $t_0\in(t_1,t_2)$, $(Ay_1)(t)<0$ for all $t\in(t_1,t_2)$, and $(Ay_1)(t_1)=(Ay_1)(t_2)=0$. It is easy to see that $[t_1,t_2]\neq[0,1]$ by (H_1). Without loss of generality, suppose $t_2<1$. Then $y_1(t)\equiv 0$ for $t\in[t_1,t_2]$, and $(Ay_1)(t)<0$ for $t\in(t_1,t_2)$, and $(Ay_1)(t_2)=0$. Thus, $(Ay_1)'(t_2)\geqslant 0$. (H_1) implies $(\Phi((Ay_1)'))'(t)=-q(t)f(t,0)\leqslant 0$ for $t\in(t_1,t_2)$. So, $(Ay_1)'(t)\geqslant 0$ for $t\in[t_1,t_2]$. Therefore, $t_1=0$. We get $(Ay_1)(0)\leqslant(Ay_1)(t_0)<0$, a contradiction.

We now show that (C_1) of Theorem 1 is satisfied. For $x\in\partial K'_a$, we have $\|x\|=a$. From (H_5) we obtain

$$\|T'x\| = \max_{0\leqslant t\leqslant 1/2}\left|\int_0^t \Phi^{-1}\left(\int_s^{1/2} q(u)f^*(u,x(u))\,\mathrm{d}u\right)\mathrm{d}s\right|$$
$$< \frac{1}{2}\,\Phi^{-1}\left(\int_0^{1/2} q(u)\cdot\frac{1}{\int_0^{1/2} q(t)\,\mathrm{d}t}\,\Phi(2a)\,\mathrm{d}u\right)$$
$$= \frac{1}{2}\cdot 2a = a.$$

Next, we show that (C_2) of Theorem 1 is satisfied. For $x\in\partial K'(\delta b)$, i.e., $\alpha(x)=\delta b$. For $\delta\leqslant t\leqslant 1-\delta$, we have $\delta b\leqslant x(t)\leqslant b$. we use conditions ($H_1$) and ($H_6$) to obtain

$$\alpha(T'x) = \min_{\delta\leqslant t\leqslant 1/2}\int_0^t \Phi^{-1}\left(\int_s^{1/2} q(u)f^*(u,x(u))\,\mathrm{d}u\right)\mathrm{d}s$$
$$= \int_0^{\delta} \Phi^{-1}\left(\int_s^{1/2} q(u)f^*(u,x(u))\,\mathrm{d}u\right)\mathrm{d}s$$
$$> \delta\Phi^{-1}\left(-M\int_0^{\delta} q(u)\,\mathrm{d}u + \int_{\delta}^{1/2} q(u)\cdot\frac{M\int_0^{\delta} q(t)\,\mathrm{d}t+\Phi(b)}{\int_{\delta}^{1/2} q(t)\,\mathrm{d}t}\,\mathrm{d}u\right)$$
$$= \delta b.$$

Using the continuity of f and the definition of f^*, there is $c>b$ such that $\|T'x\|<c$ for $\alpha(x)\leqslant b$. Applying Theorem 1, T' has a fixed point y_2 such that $y_2\in K'_a(\delta b)$.

Finally, we show that $Ax=T'x$ for $x\in K'_a(\delta b)\cap\{u : T'u=u\}$. Let $x\in K'_a(\delta b)\cap\{u : T'u=u\}$, then

$$\|x\| > a \geqslant \frac{1-\delta}{2}\Phi^{-1}\left[\Phi\left(\frac{2d}{\delta}\right)+M\int_0^{\delta/2} q(t)\,\mathrm{d}t\right]+d.$$

We claim $\|x\| = \max_{\delta/2 \leqslant t \leqslant 1-\delta/2} x(t)$. If there is $t_0 \in (0, \delta/2)$ such that $x(t_0) = \|x\| > a$, then

$$x'(t_0) = (A'x)'(t_0) = \Phi^{-1}\left(\int_{t_0}^{1/2} q(u) f^*(u, x(u))\,\mathrm{d}u\right) = 0,$$

i.e.,

$$\int_{t_0}^{1/2} q(u) f^*(u, x(u))\,\mathrm{d}u = 0.$$

From (H_7), we have

$$\begin{aligned} x(t_0) = \|x\| &= \int_0^{t_0} \Phi^{-1}\left(\int_s^{1/2} q(u) f^*(u, x(u))\,\mathrm{d}u\right)\mathrm{d}s \\ &= \int_0^{t_0} \Phi^{-1}\left(\int_s^{t_0} q(u) f^*(u, x(u))\,\mathrm{d}u\right)\mathrm{d}s \\ &\leqslant \frac{\delta}{2}\Phi^{-1}\left(\int_0^{\delta/2} q(u)\cdot\frac{1}{\int_0^{\delta/2} q(t)\,\mathrm{d}t}\,\Phi\left(\frac{2a}{\delta}\right)\mathrm{d}u\right) \\ &= a, \end{aligned}$$

a contradiction. Therefore, $\|x\| = \max_{\delta/2 \leqslant t \leqslant 1-\delta/2} x(t)$.

Next we will prove $x(\delta/2) \geqslant d$. Suppose this is not true, then there exists $t_0 \in (\delta/2, \frac{1}{2})$ such that

$$x'(t_0) > \Phi^{-1}\left[\Phi\left(\frac{2d}{\delta}\right) + M\int_0^{\delta/2} q(t)\,\mathrm{d}t\right].$$

It follows from the concavity of x on $[\delta/2, 1-\delta/2]$ that

$$x\left(\frac{\delta}{2}\right) \geqslant x'(t_0) > \Phi^{-1}\left[\Phi\left(\frac{2d}{\delta}\right) + M\int_0^{\delta/2} q(t)\,\mathrm{d}t\right].$$

For $0 \leqslant t \leqslant \delta/2$, we have

$$\begin{aligned} \Phi(x'(t)) &= \Phi\left[x'\left(\frac{\delta}{2}\right)\right] - \int_t^{\delta/2} (\Phi(x'(s)))'\,\mathrm{d}s \\ &= \Phi\left[x'\left(\frac{\delta}{2}\right)\right] + \int_t^{\delta/2} q(s) f^*(s, x(s))\,\mathrm{d}s \\ &\geqslant \left[\Phi\left(\frac{2d}{\delta}\right) + M\int_0^{\delta/2} q(t)\,\mathrm{d}t\right] - M\int_0^{\delta/2} q(s)\,\mathrm{d}s \\ &= \Phi\left(\frac{2d}{\delta}\right), \end{aligned}$$

i.e., $x'(t) \geqslant 2d/\delta$. Therefore,

$$0 = x(0) = x\left(\frac{\delta}{2}\right) - \int_0^{\delta/2} x'(s)\,\mathrm{d}s < d - \frac{\delta}{2}\cdot\frac{2d}{\delta} = 0,$$

a contradiction. Thus, $d \leqslant x(t) \leqslant b$ for $\delta/2 \leqslant t \leqslant 1-\delta/2$. From the definition of f^*, we have $f^*(t,x(t)) = f(t,x(t))$ for $0 \leqslant t \leqslant 1$. Then $Ax = T'x$ for $x \in K_a'(\delta b) \cap \{u : T'u = u\}$. Thus, y_2 is a solution of (1). Theorem 2 is now completed. □

4. Mixed boundary problem

Let $X = C[0,1], P = \{x \in X : x(t) \geqslant 0 \text{ for } t \in [0,1]\}$, $P' = \{x \in P : x \text{ is concave, nondecreasing on } [\delta/2, 1]\}$, where $\delta \in (0,1)$. Obviously, $P, P' \subset X$ are two cones with $P' \subset P$.

We suppose that there are $a, b, d > 0$ satisfying

$$0 < \frac{2-\delta}{2}\Phi^{-1}\left[\Phi\left(\frac{2d}{\delta}\right) + M\int_0^{\delta/2} q(t)\,\mathrm{d}t\right] + d \leqslant a < \frac{\delta}{2-\delta}b < b$$

such that

(G_1) $f(t,x)\displaystyle\int_{\delta/2}^{1} q(t)\,\mathrm{d}t \geqslant M\int_0^{\delta/2} q(t)\,\mathrm{d}t$ for $(t,x) \in [\delta/2, 1] \times [d, b]$;

(G_2) $f(t,x)\displaystyle\int_0^1 q(t)\,\mathrm{d}t < \Phi(a)$ for $(t,x) \in [0,1] \times [0,a]$;

(G_3) $f(t,x)\displaystyle\int_{\delta}^{1} q(t)\,\mathrm{d}t \geqslant M\int_0^{\delta} q(t)\,\mathrm{d}t + \Phi(b/(2-\delta))$ for $(t,x) \in [\delta, 1] \times [\delta/(2-\delta)b, b]$;

(G_4) $f(t,x)\displaystyle\int_0^{\delta/2} q(t)\,\mathrm{d}t \leqslant \Phi\left(\frac{2a}{\delta}\right)$ for $(t,x) \in [0, \delta/2] \times [0, b]$.

For $x \in P$ we define

$$\alpha(x) = \min_{\delta \leqslant t \leqslant 1} x(t);$$

$$(T_1x)(t) = \left[\int_0^t \Phi^{-1}\left(\int_s^1 q(u)F(u,x(u))\mathrm{d}u\right)\,\mathrm{d}s\right]^+;$$

$$(A_1x)(t) = \int_0^t \Phi^{-1}\left(\int_s^1 q(u)F(u,x(u))\,\mathrm{d}u\right)\,\mathrm{d}s;$$

$$(T_1'x)(t) = \int_0^t \Phi^{-1}\left(\int_s^1 q(u)f^*(u,x(u))\,\mathrm{d}u\right)\,\mathrm{d}s,$$

where

$$f^*(t,x) = \begin{cases} f(t,0), & 0 \leqslant t < \frac{\delta}{2}, -\infty < x < 0, \\ f(t,x), & \frac{\delta}{2} \leqslant t \leqslant 1, d \leqslant x \leqslant b, \\ f(t,d), & \frac{\delta}{2} \leqslant t \leqslant 1, -\infty < x < d, \\ f(t,x), & 0 \leqslant t < \frac{\delta}{2}, 0 \leqslant x \leqslant b, \\ f(t,b), & 0 \leqslant t \leqslant 1, b < x < +\infty. \end{cases}$$

Essentially the same reasoning as in Lemma 2, we can obtain $T_1 : P \to P$ and $T_1' : P' \to P'$ are completely continuous. For $x \in P'$, we have $x(t) \geqslant \delta/(2-\delta)\max_{\delta/2 \leqslant t \leqslant 1} x(t)$ for $t \in [\delta, 1]$ by the concavity of x on $[\delta/2, 1]$. Thus, $\alpha(x) \leqslant \|x\|$ and $\alpha(x) \geqslant \delta/(2-\delta)\max_{\delta/2 \leqslant t \leqslant 1} x(t)$.

Theorem 3. *Suppose* (H_1)–(H_2), (G_1)–(G_4) *hold. Then the boundary value problem* (2) *has at least two positive solutions* y_1 *and* y_2 *such that*:

$$0 < \|y_1\| < a < \|y_2\|, \min_{t \in [\delta, 1]} y_2(t) < \frac{\delta}{2-\delta} b.$$

Proof. At first we show that T_1 has a fixed point $y_1 \in P$ with $0 < \|y_1\| < a$. In fact, for all $x \in \partial P_a$, we have $\|x\| = a$. From (G_2) we obtain

$$\|T_1 x\| = \max_{0 \leqslant t \leqslant 1} \left| \left[\int_0^t \Phi^{-1} \left(\int_s^1 q(u) F(u, x(u))\,\mathrm{d}u \right) \mathrm{d}s \right]^+ \right|$$

$$< \Phi^{-1} \left(\int_0^1 q(u) \cdot \frac{1}{\int_0^1 q(t)\,\mathrm{d}t} \Phi(a)\,\mathrm{d}u \right) = a.$$

The existence of y_1 is proved by using the Schauder fixed point theorem.

Next we will prove that y_1 is a fixed point of A_1. Suppose the contrary, then there exists $t_0 \in (0,1)$ such that $y_1(t_0) \neq (A_1 y_1)(t_0)$. It must be $(A_1 y_1)(t_0) < 0 = y_1(t_0)$. Let (t_1, t_2) be the maximal interval such that $t_0 \in (t_1, t_2)$, $(A_1 y_1)(t) < 0$ for all $t \in (t_1, t_2)$. It is easy to see that $[t_1, t_2] \neq [0,1]$ by (H_1). If $t_2 < 1$, then $y_1(t) \equiv 0$ for $t \in [t_1, t_2]$, and $(A_1 y_1)(t) < 0$ for $t \in (t_1, t_2)$, and $(A_1 y_1)(t_2) = 0$. Thus, $(A_1 y_1)'(t_2) \geqslant 0$. (H_1) implies $(\Phi((A_1 y_1)'))'(t) = -q(t) f(t,0) \leqslant 0$ for $t \in (t_1, t_2)$. So, $(A_1 y_1)'(t) \geqslant 0$ for $t \in [t_1, t_2]$. We obtain $t_1 = 0$. Therefore, $(A_1 y_1)(0) \leqslant (A_1 y_1)(t_0) < 0$, a contradiction. If $t_1 > 0$, then $y_1(t) \equiv 0$ for $t \in [t_1, t_2]$, and $(A_1 y_1)(t) < 0$ for $t \in (t_1, t_2)$, and $(A_1 y_1)(t_1) = 0$. Thus, $(A_1 y_1)'(t_1) \leqslant 0$. (H_1) implies $(\Phi((A_1 y_1)'))'(t) = -q(t) f(t,0) \leqslant 0$ for $t \in (t_1, t_2)$. So, $(A_1 y_1)'(t) \leqslant 0$ for $t \in [t_1, t_2]$. We obtain $t_2 = 1$. On the other hand, $(A_1 y_1)'(1) = 0$, a contradiction.

We now show that (C_1) of Theorem 1 is satisfied. For $x \in \partial P_a'$, we have $\|x\| = a$. From (G_2) we obtain

$$\|T_1' x\| = \max_{0 \leqslant t \leqslant 1} \left| \int_0^t \Phi^{-1} \left(\int_s^1 q(u) f^*(u, x(u))\,\mathrm{d}u \right) \mathrm{d}s \right|$$

$$< \Phi^{-1} \left(\int_0^1 q(u) \cdot \frac{1}{\int_0^1 q(t)\,\mathrm{d}t} \Phi(a)\,\mathrm{d}u \right) = a.$$

Next we show that (C_2) of Theorem 1 is satisfied. For $x \in \partial P'(\delta/(2-\delta)b)$, i.e., $\alpha(x) = \delta/(2-\delta)b$. For $\delta \leqslant t \leqslant 1$, we have $\delta/(2-\delta)b \leqslant x(t) \leqslant b$. We use conditions (H_1) and (G_3) to obtain

$$\alpha(T_1' x) = \min_{\delta \leqslant t \leqslant 1} \int_0^t \Phi^{-1} \left(\int_s^1 q(u) f^*(u, x(u))\,\mathrm{d}u \right) \mathrm{d}s$$

$$= \int_0^\delta \Phi^{-1} \left(\int_s^1 q(u) f^*(u, x(u))\,\mathrm{d}u \right) \mathrm{d}s$$

$$> \delta\Phi^{-1}\left[-M\int_0^\delta q(u)\,\mathrm{d}u + \int_\delta^1 q(u)\cdot\frac{M\int_0^\delta q(t)\,\mathrm{d}t + \Phi(b/(2-\delta))}{\int_\delta^1 q(t)\,\mathrm{d}t}\mathrm{d}u\right]$$

$$= \frac{\delta}{2-\delta}b.$$

Using the continuity of f and the definition of f^*, there is $c > b$ such that $\|T_1'x\| < c$ for $\alpha(x) \leqslant b$. Applying Theorem 1, T_1' has a fixed point y_2 such that $y_2 \in P_a'(\delta/(2-\delta)b)$.

Finally, we show that $A_1x = T_1'x$ for $x \in P_a'(\delta/(2-\delta)b) \cap \{u : T'u = u\}$. Let $x \in P_a'(\delta/(2-\delta)b) \cap \{u : T_1'u = u\}$, then

$$\|x\| > a \geqslant \frac{2-\delta}{2}\Phi^{-1}\left[\Phi\left(\frac{2d}{\delta}\right) + M\int_0^{\delta/2} q(t)\,\mathrm{d}t\right] + d.$$

We claim $\|x\| = \max_{\delta/2 \leqslant t \leqslant 1} x(t)$. If there is $t_0 \in (0, \delta/2)$ such that $x(t_0) = \|x\| > a$, then

$$x'(t_0) = (A_1'x)'(t_0) = \Phi^{-1}\left(\int_{t_0}^1 q(u) f^*(u, x(u))\,\mathrm{d}u\right) = 0,$$

i.e.,

$$\int_{t_0}^1 q(u) f^*(u, x(u))\,\mathrm{d}u = 0.$$

From (G_4), we have

$$\begin{aligned} x(t_0) = \|x\| &= \int_0^{t_0} \Phi^{-1}\left(\int_s^1 q(u) f^*(u, x(u))\,\mathrm{d}u\right)\mathrm{d}s \\ &= \int_0^{t_0} \Phi^{-1}\left(\int_s^{t_0} q(u) f^*(u, x(u))\,\mathrm{d}u\right)\mathrm{d}s \\ &\leqslant \frac{\delta}{2}\Phi^{-1}\left[\int_0^{\delta/2} q(u)\cdot\frac{1}{\int_0^{\delta/2} q(t)\,\mathrm{d}t}\Phi\left(\frac{2a}{\delta}\right)\mathrm{d}u\right] \\ &= a, \end{aligned}$$

a contradiction. Therefore, $\|x\| = \max_{\delta/2 \leqslant t \leqslant 1} x(t)$.

Next we will prove $x(\delta/2) \geqslant d$. Suppose this is not true, then there exists $t_0 \in (\delta/2, 1)$ such that

$$x'(t_0) > \Phi^{-1}\left[\Phi\left(\frac{2d}{\delta}\right) + M\int_0^{\delta/2} q(t)\,\mathrm{d}t\right].$$

It follows from the concavity of x on $[\delta/2, 1]$ that

$$x'\left(\frac{\delta}{2}\right) \geqslant x'(t_0) > \Phi^{-1}\left[\Phi\left(\frac{2d}{\delta}\right) + M\int_0^{\delta/2} q(t)\,\mathrm{d}t\right].$$

For $0 \leqslant t \leqslant \delta/2$, we have

$$\begin{aligned}\Phi(x'(t)) &= \Phi\left[x'\left(\frac{\delta}{2}\right)\right] - \int_t^{\delta/2} (\Phi(x'(s)))'\,\mathrm{d}s \\ &= \Phi\left[x'\left(\frac{\delta}{2}\right)\right] + \int_t^{\delta/2} q(s) f^*(s, x(s))\,\mathrm{d}s \\ &\geqslant \left[\Phi\left(\frac{2d}{\delta}\right) + M\int_0^{\delta/2} q(t)\,\mathrm{d}t\right] - M\int_0^{\delta/2} q(s)\,\mathrm{d}s \\ &= \Phi\left(\frac{2d}{\delta}\right),\end{aligned}$$

i.e., $x'(t) \geqslant 2d/\delta$. Therefore,

$$0 = x(0) = x\left(\frac{\delta}{2}\right) - \int_0^{\delta/2} x'(s)\,\mathrm{d}s < d - \frac{\delta}{2}\cdot\frac{2d}{\delta} = 0,$$

a contradiction. Thus, $d \leqslant x(t) \leqslant b$ for $\delta/2 \leqslant t \leqslant 1$. From the definition of f^*, we have $f^*(t,x(t)) = f(t,x(t))$ for $0 \leqslant t \leqslant 1$. Then $A_1 x = T_1' x$ for $x \in P_a'(\delta/(2-\delta)b) \cap \{u : T_1' u = u\}$. So, y_2 is a solution of (2). Thus, the problem (2) has at least two positive solutions y_1 and y_2 such that

$$0 < \|y_1\| < a < \|y_2\|, \min_{t\in[\delta,1]} y_2(t) < \frac{\delta}{2-\delta} b. \qquad \square$$

Example. In problem (1), suppose that $p = 3$, $q(t) = 1$, and

$$f(t,x) = \begin{cases} 1 - 16x^2, & 0 \leqslant t \leqslant 1, 0 \leqslant x < 1/2, \\ -7 + 8x, & 0 \leqslant t \leqslant 1, 1/2 \leqslant x < 1, \\ 1 + 10(x-1)^2, & 0 \leqslant t \leqslant 1, 1 \leqslant x < 5, \\ 161 + 2500(x-5)^3, & 0 \leqslant t \leqslant 1, 5 \leqslant x < 6, \\ 2661 + 4(x-6)^2, & 0 \leqslant t \leqslant 1, x \geqslant 6. \end{cases}$$

Then (1) has at least two positive solutions.

Proof. Let $M = 3$, $d = 1$, $a = 5$, $b = 24$, $\delta = \frac{1}{4}$. Clearly, (H_1)–(H_3) hold, and $a, b, d > 0$ satisfy:

$$0 < \frac{3}{8}\Phi^{-1}\left[\Phi(8) + \frac{3}{8}\right] + 1 \leqslant 5 < 6 < 24.$$

After some simple calculation, we have

$$f(t,x) \geqslant 1 \text{ for } (t,x) \in [\tfrac{1}{8}, \tfrac{7}{8}] \times [1, 24],$$

$$f(t,x) < 200 \text{ for } (t,x) \in [0,1] \times [0,5],$$

$$f(t,y) \geqslant 2307 \text{ for } (t,x) \in [\tfrac{1}{4}, \tfrac{3}{4}] \times [6, 24],$$

$$f(t,x) \leqslant 12\,800 \text{ for } (t,x) \in [0, \tfrac{1}{8}] \times [0, 24].$$

Thus, by an application of Theorem 2, we get that problem (1) has at least two positive solutions. □

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